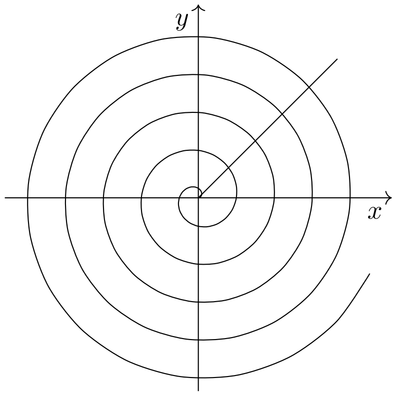
3701. Using small-angle approximations,

$$\begin{aligned} f(x) &= x \sin x \cos 2x \\ &\approx x \cdot x \cdot \left(1 - \tfrac{1}{2}(2x)^2\right) \\ &= x^2 - 2x^4. \end{aligned}$$

So, $f'(x) \approx 2x - 8x^3$ and $f''(x) \approx 2 - 24x^2$.

3702. (a) Picking e.g. $\mathbf{a} = \mathbf{i} + \mathbf{j}$, the graphs are



(b) Let $\mathbf{a} = c\mathbf{i} + d\mathbf{j}$. Then the intersections are at $p \cos p = qc$ and $p \sin p = qd$. Dividing these equations, $\tan p = d/c$. The tan function has roots every π radians, so these form an AP in p values. Squaring and adding the equations,

$$\begin{aligned} p^2 &= q^2\left(c^2 + d^2\right) \\ \therefore \quad p &= q\sqrt{c^2 + d^2}. \end{aligned}$$

The p and q values are proportional. Hence, since the p values are in arithmetic progression, so are the q values. $\square$

3703. Let $u = 1 + \sqrt{x}$. Then $du = \frac{1}{2}x^{-\frac{1}{2}}\, dx$, which gives $dx = 2\sqrt{x}\, du$ and thus $dx = 2(u-1)\, du$. Enacting the substitution,

$$\begin{aligned} &\int \frac{1}{1+\sqrt{x}}\, dx \\ &= \int \tfrac{1}{u} \cdot 2(u-1)\, du \\ &\equiv 2 \int 1 - \tfrac{1}{u}\, du \\ &\equiv 2u - 2\ln|u| + c \\ &= 2\sqrt{x} - 2\ln\left(1 + \sqrt{x}\right) + d. \end{aligned}$$

3704. By the cosine rule,

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Differentiating with respect to t,

$$\frac{d}{dt}\left(c^2\right) = 2ab \sin C \cdot \frac{dC}{dt}.$$

We know that angle C increases constantly at 1 radian per second, so $\frac{dC}{dt} = 1$. And the maximal value of $\sin C$ is 1. Hence, the maximal value of $\frac{d}{dt}\left(c^2\right)$ is $2ab$, as required.

3705. Since $x^2 \geq 0$, we know that $\sum x > 0$. Hence, $\bar{x} > 0$. Also, squaring must, on average, reduce the x values. Hence, $\bar{x} < 1$. Together, $\bar{x} \in (0, 1)$.

3706. Rearranging to make x the subject,

$$\begin{aligned} &4xy - xy^2 + 3 = 0 \\ \Longrightarrow\ &x(4y - y^2) = -3 \\ \Longrightarrow\ &x = \frac{3}{y(y-4)}. \end{aligned}$$

The denominator has roots at $y = 0$ and $y = 4$. Hence, these are the equations of the horizontal asymptotes of the curve.

3707. (a) Since the string is smooth, the tension must be the same throughout the loop. Hence, at the point where a force is pulling, the string must be symmetrical in the line of action of the pulling force. So, that line of action must be the angle bisector.

(b) By the cosine rule, the interior angles of the triangle are $(36°, 72°, 72°)$. Bisecting $72°$, the forces of magnitude F are inclined at $36°$ to the vertical. Horizontally, $2F \sin 36° = 10$, which gives $F = 5 \operatorname{cosec} 36° = 8.51$ (3sf).

3708. As far as the second, inner sum is concerned, the index i is a constant. So, we can take it out:

$$\sum_{i=1}^{10}\left(\sum_{j=1}^{10} ij\right) = \sum_{i=1}^{10}\left(i\sum_{j=1}^{10} j\right).$$

The j sum is then $1 + 2 + ... + 10 = 55$. We can take this out of the i sum as a constant factor:

$$\sum_{i=1}^{10} 55i = 55\sum_{i=1}^{10} i.$$

We can then evaluate the i sum, which is also 55. So, the total is $55^2 = 3025$.

3709. The derivative is $\frac{dy}{dx} = 2x$, so the gradient at $\left(m, m^2\right)$ is $2m$. The equation of the tangent is

$$\begin{aligned} &y - m^2 = 2m(x - m) \\ \Longrightarrow\ &y = 2mx - m^2. \end{aligned}$$

If this passes through point (a, b), then

$$\begin{aligned} &b = 2ma - m^2 \\ \Longrightarrow\ &m^2 - 2ma + b = 0 \\ \Longrightarrow\ &m = \frac{2a \pm \sqrt{4a^2 - 4b}}{2} \\ &\quad = a \pm \sqrt{a^2 - b}, \text{ as required.} \end{aligned}$$

3710. Using the conditional probability formula,

$$\begin{aligned}&\mathbb{P}(Z-\mu<\sigma \mid Z-\mu<2\sigma)\\&=\frac{\mathbb{P}(Z-\mu<\sigma)}{\mathbb{P}(Z-\mu<2\sigma)}.\end{aligned}$$

Using a calculator's normal distribution facility, the numerator and denominator are 0.8413... and 0.9772... respectively. Hence, the probability is

$$\frac{0.8413}{0.9772}=0.861\approx 86\%, \text{ as required.}$$

3711. (a) For $g^2(x)$, we apply the function twice:

$$\begin{aligned}g\big(g(x)\big)&=\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}\\&\equiv\frac{x}{(1-x)-x}\\&\equiv\frac{x}{1-2x}.\end{aligned}$$

Repeating this, $g^3(x)=\dfrac{x}{1-3x}$ etc., so

$$g^n(x)=\frac{x}{1-nx}.$$

(b) For g^n to be well defined, we require every one of $g, g^2, ..., g^n$ to be well defined. Hence, we must exclude values of x for which $1-x=0$, or $1-2x=0$, and so on. The largest possible real domain, therefore, is

$$\mathbb{R}\setminus\{1,\tfrac{1}{2},\tfrac{1}{3},...,\tfrac{1}{n}\}.$$

3712. The unit circle is $x^2+y^2=1$. Substituting into the LHS,

$$\begin{aligned}x^2+y^2&=\frac{\left(1-t^2\right)^2}{\left(1+t^2\right)^2}+\frac{4t^2}{\left(1+t^2\right)^2}\\&\equiv\frac{1-2t^2+t^4+4t^2}{1+2t^2+t^4}\\&\equiv\frac{1+2t^2+t^4}{1+2t^2+t^4}\\&\equiv 1, \text{ as required.}\end{aligned}$$

3713. Let $z=\tan x$ and $y=\cot x$. Then $\frac{dz}{dx}=\sec^2 x$ and $\frac{dy}{dx}=-\operatorname{cosec}^2 x$. Hence, by the chain rule,

$$\begin{aligned}\frac{dz}{dy}&=\frac{dz}{dx}\div\frac{dy}{dx}\\&=\sec^2 x\cdot-\sin^2 x\\&\equiv-\tan^2 x, \text{ as required.}\end{aligned}$$

3714. ① For intersections, we require

$$\begin{aligned}&x^2=\frac{1}{x^2-1}\\\Longrightarrow\ &x^4-x^2-1=0.\end{aligned}$$

This has $\Delta=5>0$. And one of its roots $x^2=\alpha$ is positive, so the curves do intersect.

② For intersections, we require

$$\begin{aligned}&x^2=\frac{1}{1-x^2}\\\Longrightarrow\ &x^4-x^2+1=0.\end{aligned}$$

This has $\Delta=-3<0$, so has no real roots. The curves do not intersect.

3715. The first ball hits the ground at $T=2$. It has speed 20 ms^{-1} at this point. Then, t seconds after the first ball hits the ground, when the second ball has been falling for $t+1$ seconds, the heights are $h_1=20t-5t^2$ and $h_2=20-5(t+1)^2$. Equating these,

$$\begin{aligned}&20t-5t^2=20-5(t+1)^2\\\Longrightarrow\ &t=\tfrac{1}{2}.\end{aligned}$$

So, the balls collide 2.5 seconds after the release of the first ball.

3716. We require

$$\begin{aligned}&\frac{x^3+3x^2}{x+2}\equiv ax^2+bx+c+\frac{d}{x+2}\\\Longrightarrow\ &x^3+3x^2=(ax^2+bx+c)(x+2)+d.\end{aligned}$$

Equating coefficients, starting with x^3, we get $a=1$, then $b=1$, $c=-2$, $d=4$. So,

$$\frac{x^3+3x^2}{x+2}\equiv x^2+x-2+\frac{4}{x+2}.$$

— Alternative Method —

Using polynomial long division,

$$\begin{array}{r|l}
 & x^2+x-2\\
x+2 & x^3+3x^2\\
 & -x^3-2x^2\\
\hline
 & x^2\\
 & -x^2-2x\\
\hline
 & -2x\\
 & 2x+4\\
\hline
 & 4
\end{array}$$

So, $\dfrac{x^3+3x^2}{x+2}\equiv x^2+x-2+\dfrac{4}{x+2}.$

3717. Let $y=f(x)$. Separating the variables,

$$\begin{aligned}&\frac{dy}{dx}=(\ln 2)y\\\Longrightarrow\ &\int\frac{1}{y}\,dy=\int\ln 2\,dx\\\Longrightarrow\ &\ln|y|=x\ln 2+c\\\Longrightarrow\ &y=Ae^{x\ln 2}\\&\equiv A\cdot 2^x.\end{aligned}$$

3718. Since $p = q$ is a root of the numerator, $(p-q)$ must be a factor. Taking it out,

$$\begin{aligned}
&\lim_{p,q\to x} \frac{p^5 - q^5}{p-q} \\
\equiv &\lim_{p,q\to x} \frac{(p-q)\left(p^4 + p^3q + p^2q^2 + pq^3 + q^4\right)}{p-q} \\
\equiv &\lim_{p,q\to x} p^4 + p^3q + p^2q^2 + pq^3 + q^4 \\
\equiv &\ 5x^4.
\end{aligned}$$

— Nota Bene —

This is an alternative form of differentiation from first principles (of x^5 in this case), in which two points *both* tend towards x. We assume that p and q are distinct.

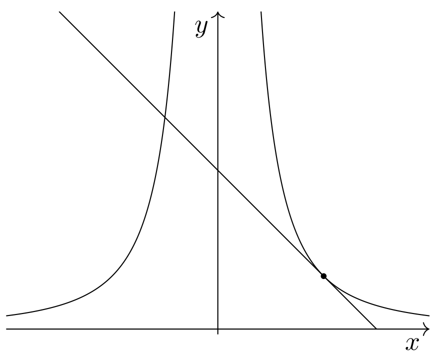
3719. (a) The least possible value of $\mathrm{f}(x) + \mathrm{g}(x)$ is $a + c$, and its greatest possible value is $b + d$. Hence, the smallest set which can be guaranteed to contain the range of h is $[a + c, b + d]$.

(b) The least possible value of $\mathrm{f}(x) - \mathrm{g}(x)$ is $a - d$ and its greatest possible value is $b - c$. Hence, the smallest set which can be guaranteed to contain the range of h is $[a - d, b - c]$.

3720. (a) Raising base and input to the same power, $\log_2 x \equiv \log_4 x^2$. So, we can simplify as

$$\begin{aligned}
&\log_2 x + \log_4 y = 1 \\
\Longrightarrow\ &\log_4 x^2 + \log_4 y = 1 \\
\Longrightarrow\ &\log_4(x^2y) = 1 \\
\Longrightarrow\ &x^2y = 4.
\end{aligned}$$

This is a squared reciprocal graph $y = \frac{4}{x^2}$. With a line of gradient -1 which is tangent to the curve, the curves are



(b) We need a gradient of -1. Differentiating,

$$\frac{dy}{dx} = -8x^{-3}.$$

So, we require $-8x^{-3} = -1$, which gives the point of tangency as $(2, 1)$. Hence, $k = 3$.

3721. For a conditioning approach, place the first tile wlog. For the 2×2 square to have symmetry order 2, the diagonally opposite tile must match. This has probability $1/4$.

Consider the remaining tiles. These must match each other, and they must not match the first two. The probability of success for the third tile is $3/4$, and for the fourth tile is $1/4$. So, the probability p of rotational symmetry order 2 is

$$p = \tfrac{1}{4} \cdot \tfrac{3}{4} \cdot \tfrac{1}{4} = \tfrac{3}{64}.$$

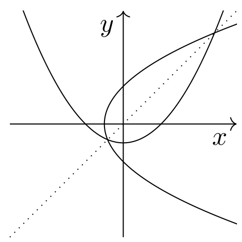
— Alternative Method —

For a combinatorial approach, the possibility space contains 4^4 outcomes. For symmetry order 4, there are 4 outcomes. For symmetry order 2 (including order 4), there are 4^2 outcomes. So, for order 2 (excluding order 4),

$$p = \frac{4^2 - 4}{4^4} = \tfrac{3}{64}.$$

3722. Consider $x \to k^-$. Approaching k from below, the gradient is positive and is increasing. The gradient cannot, therefore, become 0 at $x = k$. Hence, the statement is false.

3723. The parabolae are reflections in the line $y = x$:



So, they intersect exactly where they intersect with $y = x$. Hence, we need p such that the first parabola and $y = x$ do not intersect. The equation for intersections is

$$\begin{aligned}
&\tfrac{1}{4}x^2 + p = x \\
\Longrightarrow\ &\tfrac{1}{4}x^2 - x + p = 0.
\end{aligned}$$

We need $\Delta < 0$, which is $1 - p < 0$, so $p \in (1, \infty)$.

3724. We rewrite the function as

$$\begin{aligned}
\mathrm{f}(x) &= \frac{x^2 + 8x + 9}{x^2 + 8x - 9} \\
&\equiv \frac{x^2 + 8x - 9 + 18}{x^2 + 8x - 9} \\
&\equiv 1 + \frac{18}{x^2 + 8x - 9} \\
&\equiv 1 + \frac{18}{(x+4)^2 - 25}.
\end{aligned}$$

The range of the denominator is $[-25, \infty)$. So, the range of the fraction is $\left(-\infty, -18/25\right] \cup \left(0, \infty\right)$. This gives the range of f as $\left(-\infty, 7/25\right] \cup \left(1, \infty\right)$.

3725. The derivative is $3x^2 - 1$. So, at $(a, a^3 - a)$, the gradient of the cubic is $m = 3a^2 - 1$. The equation of the tangent is

$$y - (a^3 - a) = (3a^2 - 1)(x - a).$$

Substituting $x = a - 6$ and $y = (a - 6)^3 - (a - 6)$,

$$\begin{aligned} &(a-6)^3 - (a-6) - (a^3 - a) \\ &\qquad = (3a^2 - 1)(a - 6 - a) \\ \Longrightarrow\ & -18a^2 + 108a - 210 = -18a^2 + 6 \\ \Longrightarrow\ & a = 2. \end{aligned}$$

3726. Subtracting the equations, the terms in y and z terms cancel. This leaves

$$\begin{aligned} & x^2 - (x-1)^2 = 0 \\ \Longrightarrow\ & x = \tfrac{1}{2}. \end{aligned}$$

So, the intersection lies within the plane $x = \frac{1}{2}$. Substituting this back into the equation of S_1,

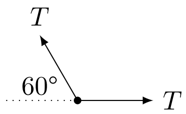
$$\begin{aligned} & \tfrac{1}{4} + y^2 + z^2 = 1 \\ \Longrightarrow\ & y^2 + z^2 = \tfrac{3}{4}. \end{aligned}$$

This is a (y, z) circle, centred on $(0, 0)$ in the (y, z) plane with radius $\sqrt{3}/2$. Its centre in (x, y, z) space is $(1/2, 0, 0)$.

——— Nota Bene ———

For additional visualisation, you might well find a 3D graphing calculator such as Desmos helpful.

3727. (a) Consider only the lower bank. Horizontally, the forces cancel on all but the right-most peg. For that peg, the forces are



Resolving horizontally, the resultant force is $T - T\cos 60° \equiv \frac{1}{2}T$.

(b) In the sections of string from peg to peg, the vertical component of tension is $T \sin 60°$. There are $2n - 1$ such strands. The resultant force, therefore, is

$$(2n - 1) \times T \sin 60° \equiv \left(n - \tfrac{1}{2}\right)\sqrt{3}T.$$

3728. To stretch by scale factor 3 in the y direction, we replace y by $\frac{1}{3}y$, giving the new equation as

$$\mathrm{f}(x) + \mathrm{f}\left(\tfrac{1}{3}y\right) = 1.$$

3729. Using identities, the $\cot\left(\theta - \frac{\pi}{2}\right) \equiv -\tan\theta$. Quoting the small-angle approximation for tan, this gives $\cot(\theta - \frac{\pi}{2}) \approx -\theta$, for small θ in radians.

——— Alternative Method ———

Let $y = \cot\left(\theta - \frac{\pi}{2}\right)$. Then the derivative is

$$\frac{dy}{d\theta} = -\operatorname{cosec}^2\left(\theta - \tfrac{\pi}{2}\right).$$

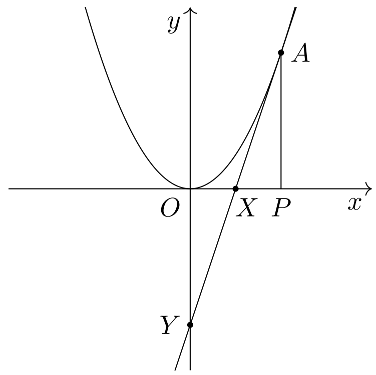
At $\theta = 0$, $\frac{dy}{dx} = 0$ and $y = 0$. Hence, the equation of the tangent at $\theta = 0$ is $y = -\theta$, so $\cot\left(\theta - \frac{\pi}{2}\right) \approx -\theta$ for small θ in radians.

3730. Call the *squared* edge lengths a, b, c. We require

$$\begin{aligned} a + b &= 500^2 \\ b + c &= 707^2 \\ c + a &= 843^2. \end{aligned}$$

Eliminating c from the second and third equations gives $a - b = 843^2 - 707^2 = 210800$. Solving with the first gives $a = 230400$ and $b = 19600$. So, $c = 480249$. Hence, the edges of the cuboid have lengths $(480, 140, 693)$.

3731. (a) The scenario is



Triangles APX and OXY are similar, since they contain opposite and right angles. We need to show that $|AP| = |OY|$. The equation of the tangent at $\left(a, a^2\right)$ is

$$y - a^2 = 2a(x - a).$$

Substituting $x = 0$ gives $y = -a^2$. Hence,

$$|OY| = a^2 = |AP|.$$

So, the triangles are congruent, as required.

3732. Let $y = \ln x$, so $x = e^y$. Differentiating implicitly with respect to x,

$$\begin{aligned} & 1 = e^y \frac{dy}{dx} \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{1}{e^y} \\ & \qquad = x^{-1}, \text{ as required.} \end{aligned}$$

3733. (a) Differentiating with respect to t,

$$\begin{aligned} y &= (7t-2)e^{-t} - \cos t \\ \Longrightarrow \dot{x} &= 7e^{-t} - (7t-2)e^{-t} + \sin t \\ &\equiv (-7t+9)e^{-t} + \sin t \\ \Longrightarrow \ddot{x} &= -7e^{-t} - (-7t+9)e^{-t} + \cos t \\ &\equiv (7t-16)e^{-t} + \cos t. \end{aligned}$$

Substituting these into the LHS of the DE, the exponential terms sum to zero:

$$(7t-16)e^{-t} + 2(-7t+9)e^{-t} + (7t-2)e^{-t} \equiv 0.$$

The trig terms are $\cos t + 2\sin t - \cos t$, which is $2\sin t$ as required on the RHS.

(b) Setting the velocity $\dot{x}$ to zero, we need to solve $(-7t+9)e^{-t} + \sin t = 0$. This is not analytically solvable. The N-R iteration is

$$t_{n+1} = t_n - \frac{(-7t_n+9)e^{-t_n} + \sin t_n}{(7t_n-16)e^{-t_n} + \cos t_n}.$$

We want the first t value, so we choose $t_0 = 0$. Running the iteration, $t_1 = 0.6$, after which $t_n \to 2.35147...$ So, $t = 2.35$ (3sf).

(c) As $t \to \infty$, the decay e^{-t} dominates the linear factor $(7t-2)$, and sends the first term to zero. Hence, the displacement approaches sinusoidal oscillation: $x = -\cos t$.

3734. (a) Factorising the difference of two squares,

$$\begin{aligned} &\lim_{a\to x} \frac{a-x}{a^2-x^2} \\ \equiv\ &\lim_{a\to x} \frac{a-x}{(a-x)(a+x)} \\ \equiv\ &\lim_{a\to x} \frac{1}{a+x} \\ \equiv\ &\frac{1}{2x}. \end{aligned}$$

(b) Factorising the difference of two cubes,

$$\begin{aligned} &\lim_{a\to x} \frac{a-x}{a^3-x^3} \\ \equiv\ &\lim_{a\to x} \frac{a-x}{(a-x)\left(a^2+ax+x^2\right)} \\ \equiv\ &\lim_{a\to x} \frac{1}{a^2+ax+x^2} \\ \equiv\ &\frac{1}{3x^2}. \end{aligned}$$

3735. (a) $\displaystyle\int_a^b \mathrm{f}(x)\,dx = \Big[\mathrm{F}(x)\Big]_a^b = q-p.$

(b) $\displaystyle\int_{a+1}^{b+1} \mathrm{f}(x-1)\,dx = \Big[\mathrm{F}(x-1)\Big]_{a+1}^{b+1} = q-p.$

(c) $\displaystyle\int_{3a}^{3b} \mathrm{f}\big(\tfrac{1}{3}x\big)\,dx = \Big[3\,\mathrm{F}\big(\tfrac{1}{3}x\big)\Big]_{3a}^{3b} = 3q-3p.$

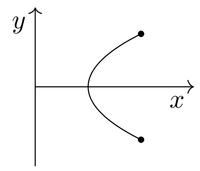
3736. Squaring both sides,

$$\begin{aligned} &\sqrt{x+2y} + \sqrt{x-2y} = 2 \\ \Longrightarrow\ &x+2y+2\sqrt{(x+2y)(x-2y)} + x - 2y = 4 \\ \Longrightarrow\ &2\sqrt{x^2-4y^2} = 4-2x \\ \Longrightarrow\ &4x^2 - 16y^2 = (4-2x)^2 \\ \Longrightarrow\ &x = y^2+1 \end{aligned}$$

This is parabolic, as required.

— Nota Bene —

The original graph is the part of this parabola for which both square roots are defined, $y \in [-1, 1]$:



3737. Let $t = 2\theta$. So $dt = 2\,d\theta$. This gives

$$\begin{aligned} &\int_0^{2\pi} \sqrt{2-2\cos t}\,dt \\ =\ &\int_0^{\pi} 2\sqrt{2-2\cos 2\theta}\,d\theta. \end{aligned}$$

Using a double-angle formula, this is

$$\begin{aligned} &\int_0^{\pi} 2\sqrt{4\sin^2\theta}\,d\theta \\ =\ &\int_0^{\pi} 4\sin\theta\,d\theta \\ =\ &\Big[-4\cos\theta\Big]_0^{\pi} \\ =\ &8, \text{ as required.} \end{aligned}$$

3738. For the tangents to cross the curve and intersect it only once, the points of tangency must be points of inflection. Using the quotient rule,

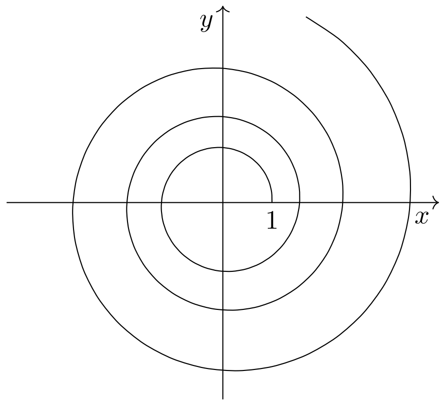
$$\begin{aligned} y &= \frac{8}{1+x^2} \\ \Longrightarrow \frac{dy}{dx} &= \frac{-16x}{(1+x^2)^2} \\ \Longrightarrow \frac{d^2y}{dx^2} &= \frac{16(3x^2-1)}{(1+x^2)^2}. \end{aligned}$$

The second derivative has single roots $x = \pm\sqrt{3}/3$, so these are points of inflection. The curve has even symmetry, so we need only analyse one of them. The gradient at $(\sqrt{3}/3, 6)$ is $-3\sqrt{3}$. Hence, the equation of the tangent is

$$\begin{aligned} y - 6 &= -3\sqrt{3}\left(x - \tfrac{\sqrt{3}}{3}\right) \\ \Longrightarrow y &= -3\sqrt{3}x + 9. \end{aligned}$$

By symmetry, both tangents intersect the y axis, and therefore each other, at $y = 9$. □

3739. (a) The equations $x = \cos t$, $y = \sin t$ define a point rotating anticlockwise around the origin, at radius 1. Scaling the radius by ae^{kt}, which is exponential growth, produces a spiral:



(b) Using the product rule,

$$\begin{aligned}\frac{dx}{dt} &= ake^{kt}\cos t - ae^{kt}\sin t\\ &\equiv a(k\cos t - \sin t)e^{kt}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{dy}{dt} &= ake^{kt}\sin t + ae^{kt}\cos t\\ &\equiv a(k\sin t + \cos t)e^{kt}.\end{aligned}$$

At points with gradient 1, the derivatives with respect to t are equal. So,

$$\begin{aligned}& a(k\sin t + \cos t)e^{kt} = a(k\cos t - \sin t)e^{kt}\\ \Longrightarrow\ & k\sin t + \cos t = k\cos t - \sin t\\ \Longrightarrow\ & \sin t(k+1) = \cos t(k-1)\\ \Longrightarrow\ & \tan t = \frac{k-1}{k+1}, \text{ as required.}\end{aligned}$$

3740. The values of $\sum x$ are $63.2 \times 100 = 6320$ for the whole sample, $64.8 \times 20 = 1296$ for the set of 20, and therefore $6320 - 1296 = 5024$ for the 80 left. So, the new mean is $\frac{5024}{80} = 62.8$.

For the standard deviation, we use

$$\begin{aligned}& s^2 = \tfrac{1}{n}\sum x^2 - \bar{x}^2\\ \Longrightarrow\ & \sum x^2 = n\big(s^2 + \bar{x}^2\big).\end{aligned}$$

The values of $\sum x^2$ are $100(8.1^2 + 63.2^2) = 405985$ for the whole sample, $20(7.6^2 + 64.8^2) = 85136$ for the set of 20, and so $405985 - 85136 = 320849$ for the 80 left. The new standard deviation is

$$\begin{aligned}s &= \sqrt{\frac{320849 - 80 \cdot 62.8^2}{80}}\\ &= 8.17144...\\ &= 8.2 \text{ (1dp)}.\end{aligned}$$

3741. Using $F = ma$, the acceleration is given by

$$a = \tfrac{1}{5}(24\sin 2t - 7\cos 2t).$$

The two sinusoids have the same frequency, so we can write them in harmonic form. We only need to know the amplitude, which is

$$R = \sqrt{7^2 + 24^2} = 25.$$

So, the maximum value of the magnitude of the acceleration is $\frac{1}{5} \times 25 = 5$ ms^{-2}.

3742. Only the first two terms of each expansion matter. These are $x^m + mx^{m-1}$ and $x^n - nx^{-1}$. So, we require

$$\big(x^m + mx^{m-1}\big)\big(x^n - nx^{-1}\big) \equiv x^7 - 3x^6 + ...$$

From the index of the leading term, $m + n = 7$. The coefficients of x^6 require $m - n = -3$. Solving these, $m = 2$ and $n = 5$.

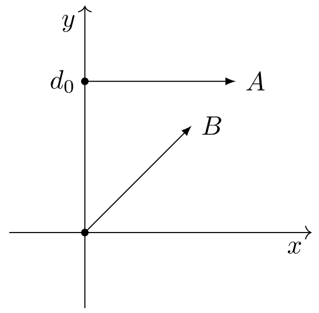
3743. (a) The line has an endpoint because $\sqrt{x+y}$ is only defined for $x + y \geq 0$. Setting $x + y = 0$ gives $x - 2(-x) = 1$. So, the endpoint is $(1/3, -1/3)$.

(b) Differentiating implicitly with respect to x,

$$\begin{aligned}& x - 2y + \sqrt{x+y} = 1\\ \Longrightarrow\ & 1 - 2\frac{dy}{dx} + \tfrac{1}{2}(x+y)^{-\frac{1}{2}}\left(1 + \frac{dy}{dx}\right) = 0\\ \Longrightarrow\ & \frac{dy}{dx}\left(\tfrac{1}{2}(x+y)^{-\frac{1}{2}} - 2\right) = -\tfrac{1}{2}(x+y)^{-\frac{1}{2}} - 1\\ \Longrightarrow\ & \frac{dy}{dx} = \frac{1 + \frac{1}{2}(x+y)^{-\frac{1}{2}}}{2 - \frac{1}{2}(x+y)^{-\frac{1}{2}}}\\ & \qquad \equiv \frac{2\sqrt{x+y} + 1}{4\sqrt{x+y} - 1}, \text{ as required.}\end{aligned}$$

(c) Substituting $(1/3, -1/3)$ gives $\frac{dy}{dx} = -1$. Hence, the equation of the tangent (one-sided) at the endpoint is $x + y = 0$.

3744. (a) Set up axes as follows:



At time t, the position vectors are

$$\mathbf{r}_A = \begin{pmatrix} ut \\ d_0 \end{pmatrix}, \quad \mathbf{r}_B = \begin{pmatrix} ut\cos 45° \\ ut\sin 45° \end{pmatrix}.$$

So, the squared distance is

$$\begin{aligned}d^2 &= (ut - ut\cos 45°)^2 + (d_0 - ut\sin 45°)^2\\ &\equiv \big(2 - \sqrt{2}\big)u^2t^2 - utd_0\sqrt{2} + d_0^2.\end{aligned}$$

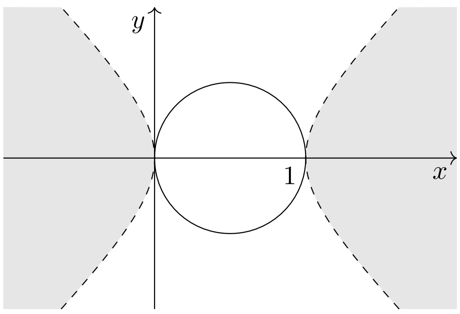
(b) Setting the rate of change of d^2 to zero,

$$\begin{aligned}
2u^2t(2-\sqrt{2}) - ud_0\sqrt{2} &= 0 \\
\implies t &= \frac{ud_0\sqrt{2}}{2u^2(2-\sqrt{2})} \\
&\equiv \frac{d_0\sqrt{2}(2+\sqrt{2})}{2u(2-\sqrt{2})(2+\sqrt{2})} \\
&\equiv \frac{d_0(1+\sqrt{2})}{2u}, \text{ as required.}
\end{aligned}$$

3745. Using a double-angle formula, the second term is $u_2 = 2\sin x\cos x$. So, the common ratio is $2\cos x$. This gives u_3 as $4\sin x\cos^2 x$. So, we solve

$$\begin{aligned}
&4\sin x\cos^2 x = 0 \\
\implies &\sin x = 0 \text{ or } \cos x = 0 \\
\therefore \quad &x = 0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2}.
\end{aligned}$$

3746. The equation $x^2 - 2rx + y^2 = 0$ defines a circle. Completing the square to $(x-r)^2 + y^2 = r^2$, this has centre $(r, 0)$ and radius r. So, it is tangent to the left-hand branch of the hyperbola at the origin. For large values of r, it intersects the RH branch and so extends into the RH shaded region. The largest value of r occurs when it is also tangent to the right-hand branch:



So, the largest value is $r_{\max} = \frac{1}{2}$.

3747. The boundary cases are maximal and minimal overlap between events A and B. With maximal overlap, A is a subset of B. In this case,

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{0.5}{0.75} = \frac{2}{3}.$$

With minimal overlap, the union of A and B is the universal set. In this case, $\mathbb{P}(A\cap B) = 0.25$, giving

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{0.25}{0.75} = \frac{1}{3}.$$

So, $\mathbb{P}(A \mid B) \in \left[{}^1\!/\!_3, {}^2\!/\!_3\right]$.

3748. This is a quadratic in $x^{\frac{1}{3}}$.

$$\begin{aligned}
&3x^{\frac{1}{3}} + 11 = 4x^{-\frac{1}{3}} \\
\implies &3x^{\frac{2}{3}} + 11x^{\frac{1}{3}} - 4 = 0 \\
\implies &(3x^{\frac{1}{3}} - 1)(x^{\frac{1}{3}} + 4) = 0 \\
\implies &x^{\frac{1}{3}} = \tfrac{1}{3}, -4 \\
\implies &x = \tfrac{1}{27}, -64.
\end{aligned}$$

3749. To translate by $2\mathbf{i} - 3\mathbf{j}$, we replace x by $x-2$ and y by $y+3$. This gives

$$\mathrm{f}(x-2) + \mathrm{g}(y+3) = 10.$$

3750. Taking out a factor of $\sqrt{x}$ from the first bracket, we have a difference of two squares. So,

$$\begin{aligned}
\mathrm{g}(x) &= \sqrt{x}(1-x) \\
&\equiv x^{\frac{1}{2}} - x^{\frac{3}{2}}.
\end{aligned}$$

For SPs, we set the first derivative to zero:

$$\tfrac{1}{2}x^{-\frac{1}{2}} - \tfrac{3}{2}x^{\frac{1}{2}} = 0.$$

Multiplying both sides by $2x^{\frac{1}{2}}$ gives $1 - 3x = 0$. So, there is a stationary point at

$$\left(\tfrac{1}{3}, -\tfrac{2\sqrt{3}}{9}\right).$$

To find the range, we must also consider the ends of the domain. At $x = 0$, $y = 0$. And, as $x \to \infty$, $y \to \infty$. So, the stationary point must be a global minimum. As required, the range is

$$\left\{y \in \mathbb{R} : y \geq -\tfrac{2\sqrt{3}}{9}\right\}.$$

3751. Four edges meet at every vertex. As the ant leaves A, it reduces the available edges at A to three. The next time it visits A it reduces this number to one. Then, to walk all the edges, it must reduce this number to zero, which requires walking back to A. Hence, its route must finish at A. QED.

3752. The quartic $ax^4 + bx^2 + c = 0$, which is a quadratic in x^2, has exactly two real roots. So, we cannot have $\Delta < 0$. $\Delta = 0$ is possible, if the single root $x^2 = k$ is positive. $\Delta > 0$ is also possible, if one of the distinct roots $x^2 = k_1, k_2$ is positive and the other negative.

(a) No. The case $\Delta = 0$ would produce one root.

(b) No. This is a quadratic in x^3, which is an odd power. So, $\Delta = 0$ would produce one root.

(c) Yes. This is a quadratic in x^4, which is an even power. This has as many roots as the original equation.

3753. The angles of inclination (between line and x axis) are α and β respectively. So, L is inclined at $\alpha-\beta$ above the mirror line. Hence, the image is inclined at $\alpha-\beta$ below the mirror line. This is $\beta-(\alpha-\beta)\equiv 2\beta-\alpha$ above the x axis.

So, the gradient of the image is $\tan(2\beta-\alpha)$, and its equation is therefore $y = x\tan(2\beta-\alpha)$.

3754. The circle has equation $x^2+(y-r)^2=r^2$. Solving for intersections, we substitute for x^2:

$$\begin{aligned} & y+(y-r)^2=r^2 \\ \Longrightarrow\; & y^2+(1-2r)y=0 \\ \Longrightarrow\; & y=0,1-2r. \end{aligned}$$

The origin is a point of intersection, irrespective of the value of r. So, we need $y=1-2r$ to produce no solution points, or to reproduce the origin. Since $y=x^2\geq 0$, this occurs when $1-2r\leq 0$. This is $r\leq 1/2$. Overall, $r\in(0,1/2]$.

3755. Solving for intersections, $x=(x^2-1)^2$, which is

$$x^4-2x^2-x+1=0.$$

Using a polynomial solver, the intersections are $(0.52489, 0.27551)$ and $(1.4902, 2.22074)$. The area enclosed is given by

$$\int_{0.27551}^{2.22074}\sqrt{y}-(y-1)^2\,dy.$$

Using the integration facility on a calculator, this is 1.38 to 3sf.

3756. The derivative of the numerator is $2\cos x-2\cos 2x$ and of the denominator is $1-\cos x$. Hence, using L'Hôpital's rule,

$$\begin{aligned} & \lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x} \\ = & \lim_{x\to 0}\frac{2\cos x-2\cos 2x}{1-\cos x}. \end{aligned}$$

Since x is tending to zero, we can use small-angle approximations. This gives

$$\begin{aligned} & \lim_{x\to 0}\frac{2\big(1-\frac{1}{2}x^2\big)-2\big(1-\frac{1}{2}(2x)^2\big)}{1-\big(1-\frac{1}{2}x^2\big)} \\ = & \lim_{x\to 0}\frac{6x^2}{x^2} \\ = & \; 6, \text{ as required.} \end{aligned}$$

— Nota Bene —

It can be instructive to see what happens if you attempt using small-angle approximations without using L'Hôpital's rule.

3757. Differentiating implicitly with respect to x,

$$\begin{aligned} & ax+by=(bx-ay)^2 \\ \Longrightarrow\; & a+b\frac{dy}{dx}=2(bx-ay)\left(b-a\frac{dy}{dx}\right). \end{aligned}$$

Substituting $x=0$, $y=0$,

$$\begin{aligned} & a+b\frac{dy}{dx}=0 \\ \Longrightarrow\; & \frac{dy}{dx}=-\frac{a}{b}. \end{aligned}$$

So, the gradient of the normal is $\frac{b}{a}$. Hence, the equation of the normal to the curve at the origin is $ax+by=0$, as required.

3758. (a) Written longhand, this is

$$\begin{aligned} & (x_1+x_2+x_3)^2 \\ \equiv\; & x_1^2+x_2^2+x_3^2+2x_1x_2+2x_2x_3+2x_3x_1. \end{aligned}$$

(b) We can rewrite the above as

$$\begin{aligned} & \left(\sum_{i=1}^{3}x_i\right)^2 \\ \equiv\; & \sum_{i=1}^{3}x_i^2+2(x_1x_2+x_2x_3+x_3x_1). \end{aligned}$$

Substituting in the given values,

$$\begin{aligned} & 3^2=5+2(x_1x_2+x_2x_3+x_3x_1) \\ \Longrightarrow\; & x_1x_2+x_2x_3+x_3x_1=2. \end{aligned}$$

3759. Multiplying up, we need

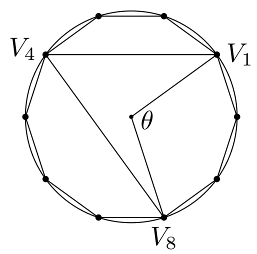
$$1\equiv A(x-6)+Bx^2.$$

The coefficient of x would require $A=0$ and of x^2 would require $B=0$. This gives 0 on the RHS. So, it is not possible to use partial fractions of this form.

Because x^2 is a repeated factor in the algebraic fraction's denominator, the correct form requires denominators x, x^2 and $x-6$:

$$\frac{1}{x^2(x-6)}\equiv\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-6}.$$

3760. Circumscribing a circle, the scenario is



The angle at the centre θ is subtended by three edges, so $\theta=\frac{3}{10}\times 360°=108°$. By the angle at the centre theorem, $\angle V_1V_4V_8=54°$.

3761. The derivatives are

$$\begin{aligned} & y = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{6}x^4 + \tfrac{1}{20}x^5 \\ \implies & \frac{dy}{dx} = 1 + x + x^2 + \tfrac{2}{3}x^3 + \tfrac{1}{4}x^4 \\ \implies & \frac{d^2y}{dx^2} = 1 + 2x + 2x^2 + x^3. \end{aligned}$$

The second derivative has a root at $x = -1$, so $(x + 1)$ is a factor. Taking this out,

$$\frac{d^2y}{dx^2} = (1 + x)(1 + x + x^2).$$

The quadratic factor has $\Delta = -3$, so it is always positive. Hence, since there is a single factor of $(1 + x)$, the second derivative is zero and changes sign at $x = -1$. This gives a point of inflection at $(-1, {}^{17}/_{60})$.

The second derivative is non-zero elsewhere, so there are no other points of inflection.

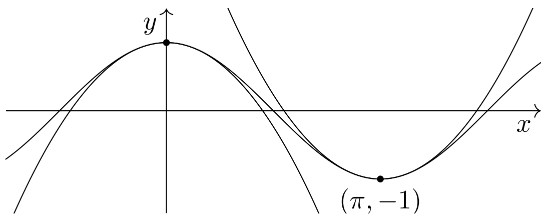
3762. (a) Using a calculator, $\mathbb{P}(X_1 > 1) = 0.159$ (3sf).

(b) The sum variable $S = X_1 + X_2 + X_3$, where $X_i \sim N(0, 1)$ are independent variables, has distribution

$$S \sim N(0, 3).$$

Using a calculator, $\mathbb{P}(S > 1) = 0.282$ (3sf).

3763. The parabola which best approximates $y = \cos x$ at $x = 0$ is $y = 1 - \frac{1}{2}x^2$. That point is a maximum. At $x = \pi$, the shape is the same, at a minimum. So, we transform the parabola $y = 1 - \frac{1}{2}x^2$ such that its vertex moves from maximum at $(0, 1)$ to minimum at $(\pi, -1)$.



This is a reflection in the x axis, combined with a translation by vector $\pi\mathbf{i}$. This gives

$$\begin{aligned} y &= -\left(1 - \tfrac{1}{2}(x - \pi)^2\right) \\ &\equiv \tfrac{1}{2}(x - \pi)^2 - 1. \end{aligned}$$

3764. Let $\theta = \arctan x$ and $\phi = \arctan y$, so $x = \tan\theta$ and $y = \tan\phi$. Using a compound-angle formula,

$$\begin{aligned} & \tan(\theta + \phi) \equiv \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \\ \therefore\quad & \theta + \phi \equiv \arctan\left(\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right) \\ \therefore\quad & \arctan x + \arctan y \equiv \arctan\left(\frac{x + y}{1 - xy}\right). \end{aligned}$$

— Nota Bene —

It is not automatically true, for *any* θ, ϕ, that

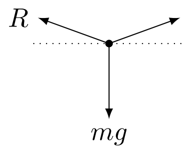
$$\arctan\big(\tan(\theta + \phi)\big) \equiv \theta + \phi.$$

It is, however, true in this question, because of the restriction $x, y \in (-1, 1)$. This restricts the angles: $\theta, \phi \in (-\pi/4, \pi/4)$. This puts $\theta + \phi \in (-\pi/2, \pi/2)$, the domain for the *invertible* tan function.

3765. There are ${}^6C_3$ equally likely ways of ordering the counters. Restricting this possibility space, there are five ways in which the red counters can end up in a group, with either $\{0, 1, 2, 3, 4\}$ blue counters to their left.

Out of these, 0 and 4 have the blue counters in a single group. So, the probability is ${}^2/_5$.

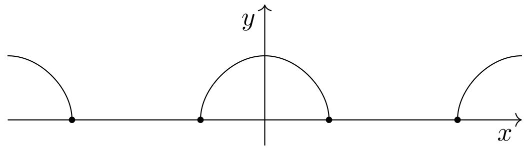
3766. Let the angle between a side of the keystone and the vertical be θ. Since half of the keystone makes up $\frac{1}{2(2k+1)}$ of the arch, $\theta = \frac{90°}{2k+1}$. Modelling the keystone as a particle and neglecting friction, the forces are



The angle between reactions and the horizontal is θ. So, resolving vertically, $2R\sin\theta = mg$, which gives

$$R = \tfrac{1}{2}mg\,\text{cosec}\left(\frac{90°}{2k + 1}\right), \text{ as required.}$$

3767. This curve is the wave $y^2 = \cos x$ with the negative parts removed. Also, since $\cos x$ has single roots, $y^2 = \cos x$ has tangents parallel to the y axis at its roots. So, the curve is



— Nota Bene —

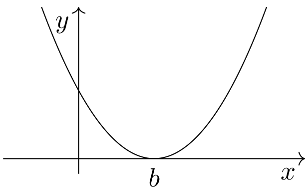
Despite appearances, the curve isn't circular.

3768. (a) This is false. Since we are taking a limit, the function g doesn't need to be defined at $x = 0$. So, a counterexample is $f(x) = 1$, $g(x) = \frac{1}{x}$. The fraction ${}^{f(x)}/_{g(x)}$ is then x, which has limit 0 as $x \to 0$. However, f doesn't tend to zero.

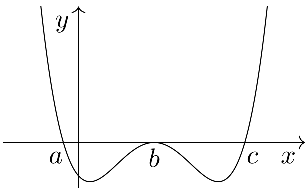
(b) This is true. Counterexamples such as the one in part (a) aren't available, because the initial equation is only satisfied if $g(0)$ is well defined. For a fraction to be zero, its numerator must be zero.

3769. The AP is symmetrical and the quadratics are monic, so the parabolae are reflections in $x = b$. So, the root at $x = b$, which both new graphs have, must be a double root, by symmetry.

(a) $y = \mathrm{f}(x) + \mathrm{g}(x)$ is a positive quadratic with a double root at $x = b$:



(b) $y = \mathrm{f}(x)\,\mathrm{g}(x)$ is a positive quartic with single roots at $x = a, c$ and a double root at $x = b$:



3770. The region common to all three circles consists of an equilateral triangle of side length 1, and three segments. Each segment subtends an angle $\frac{\pi}{3}$ at the centre. So, each segment has area

$$\begin{aligned} A_{\text{seg}} &= \tfrac{1}{2}\left(\tfrac{\pi}{3} - \sin\tfrac{\pi}{3}\right) \\ &= \tfrac{\pi}{6} - \tfrac{\sqrt{3}}{4}. \end{aligned}$$

The equilateral triangle has area $\sqrt{3}/4$, which gives the area of the shaded region as

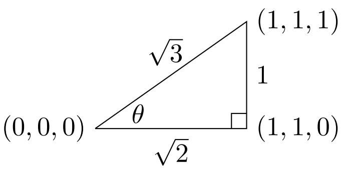
$$\begin{aligned} A &= \tfrac{\sqrt{3}}{4} + 3\left(\tfrac{\pi}{6} - \tfrac{\sqrt{3}}{4}\right) \\ &= \tfrac{1}{2}\left(\pi - \sqrt{3}\right), \text{ as required.} \end{aligned}$$

3771. (a) $x^4 - p^4 \equiv (x-p)(x+p)(x^2+p^2)$.

(b) Using this factorisation,

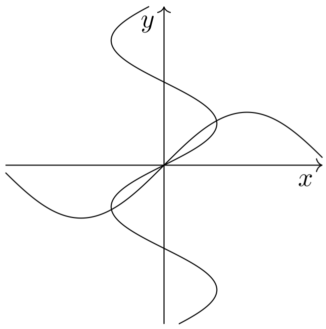
$$\begin{aligned} &\lim_{p\to x} \frac{x^4 - p^4}{x - p} \\ \equiv\ &\lim_{p\to x} \frac{(x-p)(x+p)(x^2+p^2)}{x-p} \\ \equiv\ &\lim_{p\to x} (x+p)(x^2+p^2) \\ \equiv\ &2x \cdot 2x^2 \\ \equiv\ &4x^3, \text{ as required.} \end{aligned}$$

3772. The relevant triangle has vertices $(0,0,0)$, $(1,1,0)$ and $(1,1,1)$:



So, $\theta = \arctan \frac{1}{\sqrt{2}}$.

3773. Sketching the curves (but not yet relying on the number of intersections shown), we have



Clearly, the curves intersect at $(0,0)$. To show that they intersect again (twice), we find their gradients at the origin. These are 1 for $y = \sin x$ and $1/2$ for $x = \sin 2y$. Since $1/2 < 1$, the curves must intersect again twice, as shown in the diagram. This gives exactly three points of intersection overall.

3774. The derivatives are

$$\begin{aligned} &\mathrm{f}(x) = x\ln x \\ \Longrightarrow\ &\mathrm{f}'(x) = \ln x + 1 \\ \Longrightarrow\ &\mathrm{f}''(x) = \tfrac{1}{x}. \end{aligned}$$

Substituting in,

$$\frac{\ln x + 1}{\frac{1}{x}} \equiv x\ln x + x = \mathrm{f}(x) + x.$$

So, $\mathrm{f}(x) = x\ln x$ satisfies the DE.

3775. Since $(2,5)$ is a stationary point of inflection, the cubic must take the form

$$y = k(x-2)^3 + 5.$$

Substituting in $(0,21)$ gives

$$\begin{aligned} &21 = -8k + 5 \\ \Longrightarrow\ &k = -2. \end{aligned}$$

Expanding binomially, the equation is

$$y = -2x^3 + 12x^2 - 24x + 21.$$

The coefficients are $a = -2$, $b = 12$, $c = -24$ and $d = 21$.

3776. (a) True. The first equation implies $x = 0$, which implies the second equation.

(b) True, as in part (a).

(c) True: $x^k = 0$ cannot be true for $k \leq 0$. So, if $x^k = 0$, then k must be positive, which implies that $x = 0$, which implies that $x^{k+1} = 0$.

3777. For fixed points (period 1), $\text{f}(x) = x$. So, for points fixed by $\text{f}^2(x)$ (period 2), $\text{f}^2(x) = x$. This gives

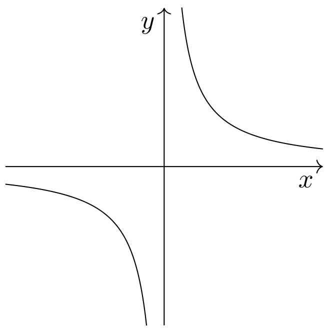
$$\begin{aligned} &\frac{1}{\frac{1}{x+1}+1} = x \\ \Longrightarrow\ &\frac{x+1}{x+2} = x \\ \Longrightarrow\ &x^2 + x - 1 = 0 \\ \Longrightarrow\ &x = \tfrac{-1\pm\sqrt{5}}{2}. \end{aligned}$$

These are the only values which could give period 2. Testing them, however, we see that

$$\begin{aligned} \text{f}\left(\tfrac{-1+\sqrt{5}}{2}\right) &= \tfrac{-1+\sqrt{5}}{2}, \\ \text{f}\left(\tfrac{-1-\sqrt{5}}{2}\right) &= \tfrac{-1-\sqrt{5}}{2}. \end{aligned}$$

So, these are, in fact, fixed points. While they do satisfy $x_{n+2} = x_n$, they do not generate period 2 behaviour.

3778. The centres of the ellipses are at $x = a$, $y = {}^1\!/_a$. Considering these as parametric equations in their own right, the locus of the centres is a standard reciprocal graph:



3779. Writing $\sin x$ as $\cos x \tan x$,

$$\begin{aligned} &6\cos x\tan x - 2\sqrt{3}\cos x - 3\tan x + \sqrt{3} = 0 \\ \Longrightarrow\ &\big(2\cos x - 1\big)\big(3\tan x - \sqrt{3}\big) = 0 \\ \Longrightarrow\ &\cos x = \tfrac{1}{2} \text{ or } \tan x = \tfrac{\sqrt{3}}{3} \\ \therefore\ \ &x = \tfrac{\pi}{6}, \tfrac{\pi}{3}, \tfrac{7\pi}{6}, \tfrac{5\pi}{3}. \end{aligned}$$

3780. (a) $\mathbb{P}(\text{all three odd}) = \frac{1}{2}^3 = \frac{1}{8}$.

(b) There are three cases here: the number of dice removed after the first roll could be $0, 1, 2$. These contribute probabilities

$$\begin{aligned} \mathbb{P}(0,3) &= \tfrac{1}{8} \times \tfrac{1}{2}^3 = \tfrac{1}{64}, \\ \mathbb{P}(1,2) &= \tfrac{3}{8} \times \tfrac{1}{2}^2 = \tfrac{3}{32}, \\ \mathbb{P}(2,1) &= \tfrac{3}{8} \times \tfrac{1}{2} = \tfrac{3}{16}. \end{aligned}$$

So, $p = \frac{1}{64} + \frac{3}{32} + \frac{3}{16} = \frac{19}{64}$.

3781. Factorising, the first equation is

$$x^2\big(2x^2 + y^4\big)\big(3x^2 - y^4\big) = 0.$$

Since $x > 0$, we reject $x = 0$. The second factor also has no real roots, since $2x^2$ and y^4 are both positive. So, $y^4 = 3x^2$. Substituting for y^4,

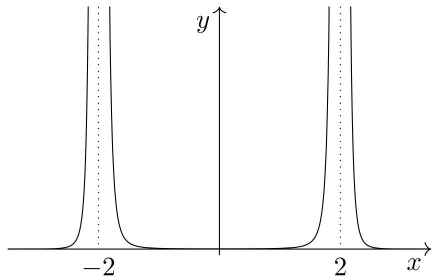
$$\begin{aligned} 1 - x^2 &= 3x^2 \\ \Longrightarrow x^2 &= \tfrac{1}{4} \\ \therefore\ \ x &= \tfrac{1}{2}. \end{aligned}$$

There is one solution point: $\left(\frac{1}{2}, \frac{\sqrt[4]{3}}{\sqrt{2}}\right)$.

3782. (a) The denominator is zero at $x = \beta$. Also, since $\text{p}(x)$ shares no common factors with $\text{q}(x)$, the numerator cannot be zero at $x = \beta$. Hence, y grows without limit as $x \to \beta$, giving a vertical asymptote.

(b) If n is even, then $\text{q}(x)$ doesn't change sign at $x = \beta$ (squared factor) and neither does $\text{p}(x)$ (no factor). For $\text{f}(x)$ to change sign, n must be odd.

(c) The graph has vertical asymptotes at $x = \pm 2$. These are quadruple asymptotes, at which the y value does not change sign. Also, $y > 0$ for all x. And, as $x \to \pm\infty$, $y \to 0$. The graph is:



3783. Place the first counter wlog, top left. A 3×3 square of successful locations for the second counter is left. Continuing in this vein, the probability that none occupy the same row or column is

$$p = 1 \times \tfrac{9}{15} \times \tfrac{4}{14} \times \tfrac{1}{13} = \tfrac{6}{455}.$$

Alternative Method

There are ${}^{16}\text{C}_4$ unordered outcomes. Of these, counting different orders, there are $16 \times 9 \times 4 \times 1$ which are successful. We divide this by 4! for the number of unordered successful outcomes. So, the probability is

$$p = \frac{16 \times 9 \times 4 \times 1}{4! \times {}^{16}\text{C}_4} = \tfrac{6}{455}.$$

Nota Bene

The two approaches taken above are *conditioning* (multiplying probabilities) and then *combinatorial* (counting outcomes).

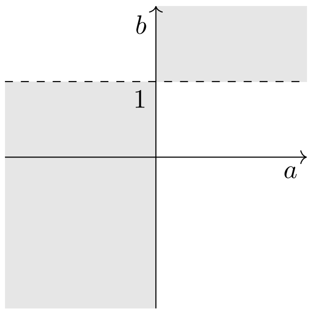
3784. Since cosine has even symmetry, the mod function has no effect. Using the first Pythagorean identity, we have a quadratic in $\cos x$:

$$\begin{aligned} & 2\sin^2 x + \cos x = 1 \\ \Longrightarrow\ & 2 - 2\cos^2 x + \cos x - 1 = 0 \\ \Longrightarrow\ & 2\cos^2 x - \cos x - 1 = 0 \\ \Longrightarrow\ & (\cos x - 1)(2\cos x + 1) = 0 \\ \Longrightarrow\ & \cos x = 1, -\tfrac{1}{2} \\ \therefore\ & x = 0, \pm\tfrac{2\pi}{3}. \end{aligned}$$

3785. The equations have no simultaneous solutions, so $x^2 + a = bx^2$ has no real roots. Rearranging, it is $x^2(1-b) = a$. So, we know that

$$\frac{a}{1-b} < 0.$$

This is satisfied if either $a < 0$ and $1 - b > 0$, or if $a > 0$ and $1 - b < 0$. On a set of (a, b) axes, this is



The b axis is not included in the region.

3786. Using a compound-angle formula,

$$\begin{aligned} & \cos\left(\tfrac{3\pi}{2} - \theta\right) \\ \equiv\ & \cos\tfrac{3\pi}{2}\cos\theta + \sin\tfrac{3\pi}{2}\sin\theta \\ \equiv\ & -\sin\theta. \end{aligned}$$

Reciprocating this identity,

$$\sec\left(\tfrac{3\pi}{2} - \theta\right) \equiv -\operatorname{cosec}\theta, \text{ as required.}$$

3787. Writing in partial fractions,

$$\begin{aligned} & \frac{a}{(x+a)(x+2a)} \equiv \frac{A}{x+a} + \frac{B}{x+2a} \\ \Longrightarrow\ & a \equiv A(x+2a) + B(x+a). \end{aligned}$$

Equating coefficients of x^1 gives $0 = A + B$, and of x^0 gives $a = 2aA + aB$, which is $1 = 2A + B$. Solving, $A = 1$ and $B = -1$. We can now integrate (no need of mod signs because $a > 0$ and $x \geq 0$):

$$\begin{aligned} & \int_0^1 \frac{1}{x+a} - \frac{1}{x+2a}\,dx \\ \equiv\ & \Big[\ln(x+a) - \ln(x+2a)\Big]_0^1 \\ \equiv\ & \ln(1+a) - \ln(1+2a) - \ln a + \ln 2a \\ \equiv\ & \ln\frac{2(1+a)}{1+2a}. \end{aligned}$$

Hence, exponentiating the original equation,

$$\begin{aligned} & \frac{2(1+a)}{1+2a} = \frac{8}{7} \\ \Longrightarrow\ & 7 + 7a = 4 + 8a \\ \Longrightarrow\ & a = 3. \end{aligned}$$

3788. We need only consider $x, y \geq 0$. So, $y = \sqrt{1-x^2}$. Substituting into $x^3 + y^3$ gives

$$x^3 + \left(1 - x^2\right)^{\frac{3}{2}}.$$

At a max, the derivative is zero:

$$\begin{aligned} & 3x^2 + \tfrac{3}{2}\cdot -2x\left(1-x^2\right)^{\frac{1}{2}} = 0 \\ \Longrightarrow\ & 3x\left(x - \sqrt{1-x^2}\right) = 0 \\ \Longrightarrow\ & x = 0 \text{ or } x = \sqrt{1-x^2} \end{aligned}$$

The latter is $x^2 = 1 - x^2$, so $x = \sqrt{2}/2$. Hence, the points for which $x^3 + y^3$ is optimised are $(0, 1)$ and $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ (and symmetrical versions.)

Testing these points, the value of $x^3 + y^3$ is 1 at $(0, 1)$ and $\sqrt{2}/2$ at $\left(\sqrt{2}/2, \sqrt{2}/2\right)$. Hence, the global maximum value is 1. □

— Alternative Method —

We need only consider $x, y \geq 0$. Since x^2 and y^2 are both positive, we know that x^2 and y^2 lie in the interval $[0, 1]$. Hence, we know that x and y lie in the interval $[0, 1]$.

For values in this interval, raising to a higher power reduces the value. This gives $x^3 \leq x^2$ and $y^3 \leq y^2$. Substituting these inequalities into $x^2 + y^2 = 1$ gives $x^3 + y^3 \leq 1$. □

3789. The horizontal and vertical *suvat*s are

$$\begin{aligned} x &= (u\cos\theta)t \\ y &= (u\sin\theta)t - \tfrac{1}{2}gt^2. \end{aligned}$$

Substituting the former into the latter,

$$\begin{aligned} y &= (u\sin\theta)\frac{x}{u\cos\theta} - \tfrac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2 \\ &\equiv x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta. \end{aligned}$$

Using the second Pythagorean trig identity,

$$\begin{aligned} & y = x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta) \\ \Longrightarrow\ & y + \frac{gx^2}{2u^2}\tan^2\theta - x\tan\theta + \frac{gx^2}{2u^2} = 0. \end{aligned}$$

3790. By the chain rule,

$$\frac{dy}{dx} = -e^{\cos x}\sin x.$$

By product and chain rules,

$$\begin{aligned}\frac{d^2y}{dx^2} &= e^{\cos x}\sin^2 x - e^{\cos x}\cos x\\ &\equiv e^{\cos x}(\sin^2 x - \cos x).\end{aligned}$$

This must be zero at a point of inflection. There is a common factor of $e^{\cos x}$, which cannot be zero. So, we divide through, leaving a quadratic in $\cos x$:

$$\begin{aligned}&\sin^2 x - \cos x = 0\\ \Longrightarrow\ &1 - \cos^2 x - \cos x = 0\\ \Longrightarrow\ &\cos^2 x + \cos x - 1 = 0\\ \Longrightarrow\ &\cos x = \tfrac{1}{2}\big(-1 \pm \sqrt{5}\big).\end{aligned}$$

The negative value of the RHS is less than -1, which means it lies outside the range of the cosine function. Hence, $\cos x = \phi$. The y coordinate at the points of inflection is therefore $k = e^{\phi}$. Taking logs, $\ln k = \phi$, as required.

3791. The second derivative is positive for all $x \in \mathbb{R}$. So, h'' must be a positive polynomial of even degree. Integrating twice, h is also a positive polynomial of even degree.

As $x \to \pm\infty$, $\text{h}(x) \to \infty$, so h must have a global minimum at $\text{h}(x) = k$, for some $k \in \mathbb{R}$. Its range, therefore, is $[k, \infty)$. □

3792. This is an infinite geometric series. With $i = 1$, the first term is $a = 4$. Addition of 1 to the index i generates multiplication by

$$\begin{aligned}r &= 2^2 \times 5^{-1}\\ &= 0.8.\end{aligned}$$

The sum to infinity, then, is

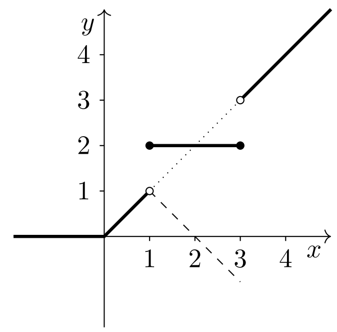
$$\begin{aligned}S_\infty &= \frac{a}{1-r}\\ &= \frac{4}{1-0.8}\\ &= 20.\end{aligned}$$

3793. This is false. A counterexample is f and g defined piecewise as follows:

$$\text{f} : \begin{cases} x \longmapsto x, & x \geq 0\\ x \longmapsto 0, & \text{otherwise,}\end{cases}$$

$$\text{g} : \begin{cases} x \longmapsto 2 - x, & 1 \leq x \leq 3\\ x \longmapsto 0, & \text{otherwise.}\end{cases}$$

These are shown below. The graphs $y = \text{f}(x)$ and $y = \text{g}(x)$ are dotted and dashed, and their sum $y = \text{f}(x) + \text{g}(x)$ is a solid line:



The intervals $[1, 2)$ and $(2, 3]$ are not in the range of $\text{f}(x) + \text{g}(x)$, which disproves the statement.

3794. Let (a, b) be the vertex in the positive quadrant. The area of the rectangle is $A = 4ab$. Then, since $b > 0$, we substitute $b = \sqrt{4 - 4a^2} \equiv 2\sqrt{1 - a^2}$. This gives

$$A = 8a\sqrt{1 - a^2}.$$

Differentiating by the product and chain rules, we set the derivative to zero and solve:

$$\begin{aligned}&8\sqrt{1 - a^2} - 4a^2\big(1 - a^2\big)^{-\frac{1}{2}} = 0\\ \Longrightarrow\ &8\big(1 - a^2\big) - 8a^2 = 0\\ \Longrightarrow\ &a = \pm\tfrac{\sqrt{2}}{2}.\end{aligned}$$

Taking the positive value, we substitute into the area formula, which gives $A = 4$. Since $A = 0$ for $a = 0$ and $a = 1$, this value is clearly a maximum. So, $A \leq 4$, as required.

3795. Writing the sum longhand, with the values $r_1 = 2$, $r_2 = 4$, $r_3 = 4$ and $r_4 = x$, we solve:

$$\begin{aligned}&\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{x}\right)^2 = 2\left(\tfrac{1}{2^2} + \tfrac{1}{4^2} + \tfrac{1}{4^2} + \tfrac{1}{x^2}\right)\\ \Longrightarrow\ &\left(1 + \tfrac{1}{x}\right)^2 = 2\left(\tfrac{3}{8} + \tfrac{1}{x^2}\right)\\ \Longrightarrow\ &1 + \tfrac{2}{x} + \tfrac{1}{x^2} = \tfrac{3}{4} + \tfrac{2}{x^2}\\ \Longrightarrow\ &\tfrac{1}{4}x^2 + 2x - 1 = 0\end{aligned}$$

So, the positive radius is $x = 2\sqrt{5} - 4$.

3796. (a) The second derivative changes sign at $x = -2$ (convex to concave) and $x = 3$ (concave to convex). It is polynomial, so $\text{f}''(-2) = 0$ and $\text{f}''(3) = 0$. Since f is quartic, f'' is quadratic. Therefore, by the factor theorem, it must take the form $\text{f}''(x) = k(x + 2)(x - 3)$ for some k. This is $\text{f}''(x) = k(x^2 - x - 6)$.

(b) Integrating the result from (a),

$$\text{f}'(x) = k\big(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 - 6x\big) + c.$$

Since $\text{f}'(0) = 0$ (*double* root), we know $c = 0$. Integrating again,

$$\text{f}(x) = k\big(\tfrac{1}{12}x^4 - \tfrac{1}{6}x^3 - 3x^2\big) + d.$$

Since $\text{f}(0) = 0$ (double *root*), we know $d = 0$. And the curve is monic, so $k = 12$. This gives $\text{f}(x) = x^4 - 2x^3 - 36x^2$.

3797. Consider the way in which the structure would fall were a girder to disappear suddenly. In particular, consider the distance between the joints between which the relevant girder was attached. If a girder $a/b/c/d$ disappeared, then the gap left would

(a) decrease, so a is in compression,

(b) increase, so b is in tension,

(c) increase, so c is in tension,

(d) decrease, so d is in compression.

— Nota Bene —

The idea of visualising what *would* happen were a physical element to disappear can be useful in a variety of scenarios. Two of these are:

① For working out whether forces are tensions (pulls) or compressions (pushes): what would happen if there was **no** force?

② For working out the direction of friction: how would the surfaces slide relative to each other if there was **no** friction?

3798. The process P_n produces values modelled by $B(n, 1/2)$. So, we can construct the possibility spaces:

(a) With P_1 and P_2, we have

	0	1	2
0	$\frac{1}{2}\cdot\frac{1}{4}$	$\frac{1}{2}\cdot\frac{2}{4}$	$\frac{1}{2}\cdot\frac{1}{4}$
1	$\frac{1}{2}\cdot\frac{1}{4}$	$\frac{1}{2}\cdot\frac{2}{4}$	$\frac{1}{2}\cdot\frac{1}{4}$

The probability that P_2 generates a greater value than P_1 is $\frac{1}{2}\cdot\frac{2}{4}+\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{2}$.

— Nota Bene —

This value $p = 1/2$ is no coincidence: the table is symmetrical, and, as shown, a symmetrical half of the entries are successes.

(b) With P_2 and P_3, we have

	0	1	2	3
0	$\frac{1}{4}\cdot\frac{1}{8}$	$\frac{1}{4}\cdot\frac{3}{8}$	$\frac{1}{4}\cdot\frac{3}{8}$	$\frac{1}{4}\cdot\frac{1}{8}$
1	$\frac{2}{4}\cdot\frac{1}{8}$	$\frac{2}{4}\cdot\frac{3}{8}$	$\frac{2}{4}\cdot\frac{3}{8}$	$\frac{2}{4}\cdot\frac{1}{8}$
2	$\frac{1}{4}\cdot\frac{1}{8}$	$\frac{1}{4}\cdot\frac{3}{8}$	$\frac{1}{4}\cdot\frac{3}{8}$	$\frac{1}{4}\cdot\frac{1}{8}$

As above, the probability is $\frac{1}{2}$.

3799. (a) False. A counterexample is $f(x) = x^2$, for which $f'(x) = 2x$ and $f'(u) = 2u = 6x + 4$.

(b) False. It is the same statement as in (a).

(c) False. Again considering $f(x) = x^2$, the LHS is $6(3x + 2)$, while the RHS is $2u = 2(3x + 2)$.

(d) True. As $f(3x+2) = f(u)$, differentiating both sides with respect to x maintains the equation.

3800. Using the product rule,

$$\frac{dy}{dx} = (\sin x + \cos x) + x(\cos x - \sin x).$$

At $x = 0$, this has value $m = 1$. So, the tangent is $y = x$. For intersections,

$$x(\sin x + \cos x) = x.$$

We are looking for intersections at $x \neq 0$, so we can divide through by x and solve by writing in harmonic form:

$$\begin{aligned} &\sin x + \cos x = 1 \\ \Longrightarrow\ &\sqrt{2}\sin\left(\theta + \tfrac{\pi}{4}\right) = 1 \\ \Longrightarrow\ &\theta + \tfrac{\pi}{4} = \tfrac{\pi}{4}, \tfrac{3\pi}{4}, ... \\ \Longrightarrow\ &\theta = 0, \tfrac{\pi}{2}, ... \end{aligned}$$

So, the tangent at the origin re-intersects the curve at $x = \frac{\pi}{2}$ (as well as infinitely many other places).

— End of 38th Hundred —